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THE ZAGREB INDICES AND SOME HAMILTONIAN PROPERTIES OF GRAPHS

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Abstract

Let $G = (V, E)$ be a graph. The first Zagreb index and second Zagreb index of G are defined as $\sum_{v \in V} d^2(v)$ and $\sum_{uv \in E} d(u)d(v)$, respectively. Using first and second Zagreb indices of graphs, we in this note present sufficient conditions for some Hamiltonian properties of graphs.

Keywords: The first Zagreb index, The second Zagreb index, Hamiltonian property

MSC: 05C09, 05C045

1. Introduction

We consider only finite undirected graphs without loops or multiple edges. Notation and terminology not defined here follow those in [2]. We use n and e to denote the number of vertices and edges of a graph, respectively. The complete graph of order n is denoted by K_n. We use G^c to denote the complement of a graph G. For a vertex v_i in a graph G, we use $d_i(G)$ to denote its degree in G. We use $\delta(G)$ to denote the minimum degree of G. We use $G \vee H$ to denote the the join of two disjoint graphs G and H. The first and second Zagreb indices were introduced by Gutman and Trinajstić in [3]. For a graph G, its first Zagreb index and second Zagreb index are defined as $Z_1(G) := \sum_{v \in V} d^2(v)$ and $Z_2(G) := \sum_{uv \in E} d(u)d(v)$, respectively. A cycle C in a graph G is called a Hamiltonian cycle of G if C contains all the vertices of G. A graph G is called Hamiltonian if G has a Hamiltonian cycle. A path P in a graph G is called a Hamiltonian path of G if P contains all the vertices of G. A graph G is called traceable if G has a Hamiltonian path.

In last several years, researchers have used different Zagreb indices to investigate the Hamiltonian properties of graphs (see [5], [1], [4]). In this note, we will present new sufficient conditions based upon the first and second Zagreb indices for the Hamiltonian and traceable graphs. The main results are as follows.

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Theorem 1. Let G be a k-connected ($k \geq 2$) graph of order n.

1) If $Z_1 \geq (n-k-1)(n^2+(k-1)n-k^2-2k)$, then G is Hamiltonian or $K_k \vee K_{k+1}^c$.

2) If $Z_2 \geq (n-1)(n-k-1)(n^2+(k-1)n-2k^2-3k)/2$, then G is Hamiltonian or $K_k \vee K_{k+1}^c$.

Theorem 2. Let G be a k-connected ($k \geq 1$) graph of order n.

1) If $Z_1 \geq (n-k-2)(n^2+kn-k^2-4k-3)$, then G is traceable or $K_k \vee K_{k+2}^c$.

2) If $Z_2 \geq (n-1)(n-k-2)(n^2+kn-2k^2-7k-5)/2$, then G is traceable or $K_k \vee K_{k+2}^c$.

2. Proofs

Proof of Theorem 1. Let G be a graph satisfying the conditions in Theorem 1. Suppose that G is not Hamiltonian. Then G is not a complete graph. We further have that $n \geq 2k+1$ otherwise $2\delta \geq 2k \geq n$ and G is Hamiltonian. Since $k \geq 2$, G contains a cycle. Choose a longest cycle C in G and give an orientation on C. Since G is not Hamiltonian, there exists a vertex $x_0 \in V(G) - V(C)$. By Menger's theorem, we can find s ($s \geq k$) pairwise disjoint (except for x_0) paths P_1, P_2, ..., P_s between x_0 and $V(C)$. Let u_i be the end vertex of P_i on C, where $1 \leq i \leq s$. We use u_i^+ to denote the successor of u_i along the orientation of C, where $1 \leq i \leq s$. Then $\{x_0, u_1^+, u_2^+, ..., u_s^+\}$ is independent otherwise G would have cycles which are longer than C. Therefore $S := \{x_0, u_1^+, u_2^+, ..., u_k^+\}$ is independent. Set $T := V(G) - S = \{v_1, v_2, ..., v_r\}$. Thus $|T| = r = n - |S| = n-(k+1) \geq k$.

Proof of 1). From the definition of Z_1, we have

$$(n-k-1)(n^2+(k-1)n-k^2-2k) \leq Z_1 = \sum_{v \in V} d^2(v)$$

$$= d^2(x_0) + d^2(u_1^+) + \cdots + d^2(u_k^+) + d^2(v_1) + \cdots + d^2(v_r)$$
$$\leq (k+1)r^2 + r(n-1)^2 = (n-k-1)(n^2+(k-1)n-k^2-2k).$$

Therefore $d(x_0) = d(u_1^+) = \cdots = d(u_k^+) = r = n-(k+1)$ and $d(v_1) = \cdots = d(v_r) = d(v_{n-(k+1)}) = n-1$. Now G is $K_r \vee K_{k+1}^c = K_{n-(k+1)} \vee K_{k+1}^c$. It is obvious that G is Hamiltonian if $r = n-(k+1) \geq (k+1)$. So it is impossible that $r \geq (k+1)$. Thus $r = n-(k+1) = k$. Therefore G is $K_k \vee K_{k+1}^c$.

Proof of 2). From the definition of Z_2, we have

$$(n-1)(n-k-1)(n^2+(k-1)n-2k^2-3k)/2 \leq Z_2$$

$$= \sum_{uv \in E} d(u)d(v) = \sum_{u \in S, v \in T, uv \in E} d(u)d(v) + \sum_{u \in T, v \in T, uv \in E} d(u)d(v)$$

$$\leq \sum_{u \in S, v \in T} d(u)d(v) + \sum_{u \in T, v \in T, u \neq v} d(u)d(v)$$

$$\leq r(n-1)(k+1)r+(n-1)(n-1)r(r-1)/2$$
$$=(n-1)(n-k-1)(n^2+(k-1)n-2k^2-3k)/2.$$

Therefore $d(x_0) = d(u_1^+) = \cdots = d(u_k^+) = r = n-(k+1)$ and $d(v_1) = \cdots = d(v_r) = d(v_{n-(k+1)}) = n-1$. Now G is $K_r \vee K_{k+1}^c = K_{n-(k+1)} \vee K_{k+1}^c$. It is obvious that G is Hamiltonian if $r = n-(k+1) \geq (k+1)$. So it is impossible that $r \geq (k+1)$. Thus $r = n-(k+1) = k$. Therefore G is $K_k \vee K_{k+1}^c$.

This completes the proof of Theorem 1.

Proof of Theorem 2. Let G be a graph satisfying the conditions in Theorem 2. Suppose that G is not traceable. Then G is not a complete graph. We further have that $n \geq 2k+2$ otherwise $2\delta \geq 2k \geq n-1$ and G is traceable. Choose a longest path P in G and give an orientation on P. Let y and z be the two end vertices of P. Since G is not traceable, there exists a vertex $x_0 \in V(G)\backslash V(P)$. By Menger's theorem, we can find s ($s \geq k$) pairwise disjoint (except for x_0) paths P_1, P_2, ..., P_s between x_0 and $V(P)$. Let u_i be the end vertex of P_i on P, where $1 \leq i \leq s$. Since P is a longest path in G, $y \neq u_i$ and $z \neq u_i$, for each i with $1 \leq i \leq s$, otherwise G would have paths which are longer than P. We use u_i^+ to denote the successor of u_i along the orientation of P, where $1 \leq i \leq s$. Then $\{x_0, y, u_1^+, u_2^+, ..., u_s^+\}$ is independent otherwise G would have paths which are longer than P. Therefore $S := \{x_0, y, u_1^+, u_2^+, ..., u_k^+\}$ is independent. Set $T := V(G) - S = \{\, v_1, v_2, ..., v_r \,\}$. Thus $|T| = r = n - |S| = n-(k+2) \geq k$.

Proof of 1). From the definition of Z_1, we have

$$(n-k-2)(n^2+kn-k^2-4k-3) \leq Z_1 = \sum_{v\in V} d^2(v)$$
$$= d^2(x_0) + d^2(y) + d^2(u_1^+) + \cdots + d^2(u_k^+) + d^2(v_1) + \cdots + d^2(v_r)$$
$$\leq (k+2)r^2 + r(n-1)^2 = (n-k-2)(n^2+kn-k^2-4k-3).$$

Therefore $d(x_0) = d(y) = d(u_1^+) = \cdots = d(u_k^+) = r = n-(k+2)$ and $d(v_1) = \cdots = d(v_r) = d(v_{n-(k+2)}) = n-1$. Now G is $K_r \vee K_{k+2}^c = K_{n-(k+2)} \vee K_{k+2}^c$. It is obvious that G is traceable if $r = n-(k+2) \geq (k+1)$. So it is impossible that $r \geq (k+1)$. Thus $r = n-(k+2) = k$. Therefore G is $K_k \vee K_{k+2}^c$.

Proof of 2). From the definition of Z_2, we have

$$(n-1)(n-k-2)(n^2+kn-2k^2-7k-5)/2 \leq Z_2$$
$$= \sum_{uv\in E} d(u)d(v) = \sum_{u\in S, v\in T, uv\in E} d(u)d(v) + \sum_{u\in T, v\in T, uv\in E} d(u)d(v)$$
$$\leq \sum_{u\in S, v\in T} d(u)d(v) + \sum_{u\in T, v\in T, u\neq v} d(u)d(v)$$
$$\leq r(n-1)(k+2)r + (n-1)(n-1)r(r-1)/2$$
$$= (n-1)(n-k-2)(n^2+kn-2k^2-7k-5)/2.$$

Therefore $d(x_0) = d(y) = d(u_1^+) = \cdots = d(u_k^+) = r = n - (k+2)$ and $d(v_1) = \cdots = d(v_r) = d(v_{n-(k+2)}) = n - 1$. Now G is $K_r \vee K_{k+2}^c = K_{n-(k+2)} \vee K_{k+2}^c$. It is obvious that G is traceable if $r = n - (k+2) \geq (k+1)$. So it is impossible that $r \geq (k+1)$. Thus $r = n - (k+2) = k$. Therefore G is $K_k \vee K_{k+2}^c$.

This completes the proof of Theorem 2.

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