MATI 3(2) (2021) 29-32 ISSN:2636-7785
http://www.matijournal.com/

Remarks on the Bounds of the First Zagreb Index of a Quadrilateral-Free Graph

Rao Li
Dept. of mathematical sciences
University of South Carolina Aiken
Aiken, SC 29801
USA

July 22, 2021

Abstract

A graph G is quadrilateral-free if G does not contain a cycle of length 4. The first Zagreb index of a graph G is defined as the sum of the squares of the vertex degrees of G. In this short note, we present upper and lower bounds for the first Zagreb index of a quadrilateral-free graph.

$Mathematics\ Subject\ Classification:\ 05C09.$

$Keywords: quadrilateral - free\ graph,\ the\ first\ Zagreb\ index.$

We consider only finite undirected graphs without loops or multiple edges. Notation and terminology not defined here follow those in [1]. Let $G = (V, E)$

be a graph with n vertices and e edges, where $V = \{ v_1, v_2, ..., v_n \}$. We assume that the vertices in G are arranged such that $\delta(G) = d_G(v_1) := d_1 \leq d_G(v_2) := d_2 \leq \cdots \leq d_G(v_n) := d_n = \Delta(G)$, where $d_G(v_i)$, for each i with $1 \leq i \leq n$, is the degree of vertex v_i in G. A graph G is quadrilateral-free if G does not contain a cycle of length 4. We define $C(x, 2)$ as $x(x-1)/2$.

The first Zagreb index was introduced by Gutman and Trinajstić in [3]. For a graph G, its first Zagreb index, denoted $M_1(G)$, is defined as $\sum_{i=1}^{n} d_G^2(v_i)$. In this short note, we will present an upper bound and a lower bound for the first Zagreb index of a quadrilateral-free graph. The main results of this note are as follows.

Theorem 1 Let G be a quadrilateral-free graph of order n with $\delta \geq 1$. Then

[1] $M_1 \leq (n - d_k)(n - d_k - 1) + d_k^2 + 3(2e - d_k) - 2(n - 1)$, for each k with $1 \leq k \leq n$.

[2] $M_1 \geq \frac{(2e-d_k-n+1)(2e-d_k-2n+2)}{n-1} + d_k^2 + 3(2e - d_k) - 2(n - 1)$, for each k with $1 \leq k \leq n$.

Proof of Theorem 1. Set $N(v_i) := \{ v : vv_i, v \in V \}$. Then $|N(v_i)| = d_i$. Fűredi established the following inequalities on Page 189 in [2].

$$C(n - \Delta, 2) = (\text{number of pairs of } (V(G) - N(v_n)))$$

$$\geq \textstyle\sum_{i=1}^{n-1} (\text{the number of pairs of } N(v_i) \cap (V(G) - N(v_n)))$$

$$\geq \sum_{i=1}^{n-1} C(|N(v_i)| - 1, 2) = \sum_{i=1}^{n-1} C(d_i - 1, 2)$$

$$\geq (n-1)C\left(\frac{\sum_{i=1}^{n-1}(d_i - 1)}{n-1}, 2\right)$$

$$= (n-1)C\left(\frac{2e - \Delta - n + 1}{n-1}, 2\right).$$

Studying Fűredi's proofs for the above inequalities in [2], we found that the proofs also imply the following inequalities. For each integer k with $1 \leq k \leq n$,

$$C(n-d_k,2) = (\text{number of pairs of } (V(G)-N(v_k)))$$
$$\geq \textstyle\sum_{i=1,i\neq k}^{n} (\text{the number of pairs of } N(v_i)\cap(V(G)-N(v_k)))$$
$$\geq \sum_{i=1,i\neq k}^{n} C(|N(v_i)|-1,2) = \sum_{i=1,i\neq k}^{n} C(d_i-1,2)$$
$$\geq (n-1)C\left(\frac{\sum_{i=1,i\neq k}^{n}(d_i-1)}{n-1},2\right)$$
$$= (n-1)C\left(\frac{2e-d_k-n+1}{n-1},2\right).$$

Now from

$$C(n-d_k,2) \geq \sum_{i=1,i\neq k}^{n} C(d_i-1,2),$$

we can derive that

$$\sum_{i=1,i\neq k}^{n} d_i^2 \leq 2C(n-d_k,2) + 3\sum_{i=1,i\neq k}^{n} d_i - 2(n-1).$$

Thus

$$M_1 = \sum_{i=1}^{n} d_i^2 \leq (n-d_k)(n-d_k-1) + d_k^2 + 3(2e-d_k) - 2(n-1).$$

This completes the proof of [1] in Theorem 1.

From

$$\sum_{i=1,i\neq k}^{n} C(d_i-1,2) \geq (n-1)C\left(\frac{2e-d_k-n+1}{n-1},2\right),$$

we can derive that

$$\sum_{i=1,i\neq k}^{n} d_i^2 \geq \frac{(2e-d_k-n+1)(2e-d_k-2n+2)}{n-1} + 3\sum_{i=1,i\neq k}^{n} d_i - 2(n-1).$$

Thus

$$M_1 = \sum_{i=1}^{n} d_i^2 \geq \frac{(2e-d_k-n+1)(2e-d_k-2n+2)}{n-1} + d_k^2 + 3(2e-d_k) - 2(n-1).$$

This completes the proof of [2] in Theorem 1.

If we let $k = 1$ in Theorem 1, then we have the following corollary.

Corollary 1 Let G be a quadrilateral-free graph of order n with $\delta \geq 1$. Then

[1] $M_1 \leq (n-\delta)(n-\delta-1) + \delta^2 + 3(2e-\delta) - 2(n-1)$.

[2] $M_1 \geq \frac{(2e-\delta-n+1)(2e-\delta-2n+2)}{n-1} + \delta^2 + 3(2e-\delta) - 2(n-1)$.

If we let $k = n$ in Theorem 1, then we have the following corollary.

Corollary 2 Let G be a quadrilateral-free graph of order n with $\delta \geq 1$. Then

[1] $M_1 \leq (n-\Delta)(n-\Delta-1) + \Delta^2 + 3(2e-\Delta) - 2(n-1)$.

[2] $M_1 \geq \frac{(2e-\Delta-n+1)(2e-\Delta-2n+2)}{n-1} + \Delta^2 + 3(2e-\Delta) - 2(n-1)$.

References

[1] J. A. Bondy and U. S. R. Murty, Graph Theory with Applications, Macmillan, London and Elsevier, New York (1976).

[2] Z. Fűredi, Graphs without quadrilaterals, Journal of Combinatorial Theory B 34 (1983) 187–190.

[3] I. Gutman and N. Trinajstić, Graph theory and molecular orbitals, total π–electron energy of alternant hydroncarbons, Chem. Phys. Lett. 17 (1972) 535–538.